# Stochastic resonance: Nonperturbative calculation of power spectra and residence-time distributions

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We examine the response of a finite-temperature two-state system to periodic driving using time-dependent transition rate theory. This system can exhibit the phenomenon of stochastic resonance, where raising the temperature increases the signal-to-noise ratio of the response. We obtain the power spectrum and the distribution of residence times nonperturbatively for any transition rates that are periodic in time. Given the drive period $T_s$, the power spectrum is the Fourier transform of the sum of "signal," which is periodic in time with period $T_s$, and "noise," which is the product of an exponential and a function periodic with period $T_s$. The residence-time distribution is the product of an exponential and a function that is periodic with period $T_s$. Both the power spectrum and the residence-time distribution can be calculated exactly given the dependence of the transition rates on the control parameter (e.g., asymmetry or temperature). We calculate the characteristics of stochastic resonance for a two-state system with activated transition rates and for a quantum-mechanical dissipative two-level system.

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## I. INTRODUCTION

Stochastic resonance (SR) is a phenomenon where the response of a nonlinear dynamical system to external driving is enhanced by the presence of noise [1–9]. The canonical example of SR consists of a particle in a double-well potential subject to both random noise (characterized by a temperature $T$) and periodic forcing, which could consist of a sinusoidal variation of the asymmetry energy $\varepsilon$ of the wells with frequency $\omega_s$ (Fig. 1). Two relevant experimental characteristics of such a system are the power spectrum describing the dynamics of the particle's location and the distribution of residence times in each well, each of which can be used as a signature of SR. SR occurs when the signal-to-noise ratio (SNR) of the power spectrum passes through a maximum as the noise level is increased [2]. In the SR regime the residence-time distribution can display a series of harmonic peaks [5,6] (in the absence of driving, it is exponential).

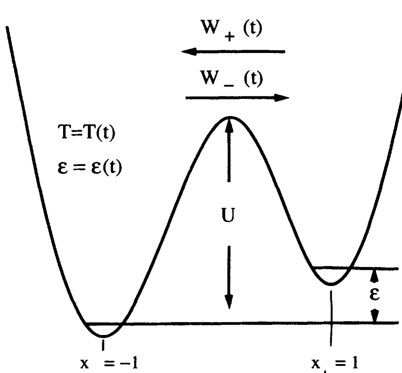


FIG. 1. Schematic diagram of asymmetric double-well potential. The transition rates $W_+$ and $W_-$ are modulated by varying the temperature $T$ or the asymmetry energy $\varepsilon$.

In this paper we discuss the power spectrum and the residence-time distribution, when the system is strongly driven. We assume that the undriven dynamics are described in terms of transition rates $W_+$ and $W_-$ between two states, and that the drive frequency and the interwell transition rates are much slower than the intrawell relaxation rate. The driven dynamics are then determined by time-dependent transition rates [2]. We show that both the power spectrum and residence-time distribution for a periodically driven two-state system possess analytic properties that allow for exact evaluation of these quantities. Even when the modulation of the transition rates is large compared to the rates themselves, exact results can be obtained straightforwardly by performing numerical integrals over a single drive period.

We calculate both the power spectrum and residence-time distribution not only for systems described by classical, thermally activated rates, but also for systems where the rates are determined by a quantum-mechanical tunneling process; we have shown previously that the quantum case can display SR [9]. The quantum case is particularly interesting because the dynamics are described accurately by rate equations even for temperatures well above the "resonance" temperature where the maximum in the signal-to-noise ratio occurs. In contrast, when the transition rates are thermally activated, the rate equation description is breaking down at the temperature of the "resonance."

We first establish some notation for the two-state system characterized by transition rates [2]. We define the probability of being in the position state $x_+=1$ $[x=-1]$ as $n_+$ $[n_-]$, and the transition rate for the system to leave that state $W_+(t)$ $[W_-(t)]$, where the time depen-

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dence of the $W$'s is induced by the external drive. We assume that the transition rates are periodic with period $T_s \equiv 2\pi/\omega_s$. Given that the system is in state $x_\pm$, the chance that it makes the transition to state $x_\mp$ in an infinitesimal time $dt$ is $W_\pm(t)dt$. Thus, the rate equation describing the population $n_+$ is

$$\frac{dn_+(t)}{dt} = W_-(t)n_-(t) - W_+(t)n_+(t) = W_-(t) - [W_+(t)+W_-(t)]n_+(t) . \quad (1.1)$$

Relevant correlation functions of $x(t)$ can be obtained from knowledge of $n_+(t)$ [2]; the power spectrum $S(\omega)$ is the Fourier transform of $C(\tau) = \langle x(t)x(t+\tau)\rangle$, where $\langle\ \rangle$ denotes the average over $t$; as shown by McNamara and Weisenfeld [2],

$$C(\tau) = \langle 2n_+(t+\tau|x_+,t)n_+(t) + 2n_+(t+\tau|x_-,t)n_+(t) - 2n_+(t) - 2n_+(t+\tau|x_-,t)+1\rangle , \quad (1.2)$$

where $n_+(t_2|x_0,t_0)$ is the probability that the system is in the state $x_+$ at time $t_2$ given that at time $t_0$ it was in state $x_0$, and

$$n_+(t) = \lim_{t_0\to-\infty} n_+(t|x_0,t_0) . \quad (1.3)$$

The power spectrum $S(\omega)$ contains a broadband noise background as well as δ-function peaks. The ratio of the coefficient of the fundamental peak and the value of the noise at $\omega_s$ is the signal-to-noise ratio (SNR).

The residence-time distribution $V_\pm(\tau)$ is the probability that the system remains in state $x_\pm$ for a duration $\tau$; it is given by

$$V_\pm(\tau) = N\int_{-\infty}^{\infty} dt_0 P_\pm(t_0+\tau|t_0)Z_\pm(t_0) , \quad (1.4)$$

where $N$ is a normalization constant, $P_\pm(t_2|t_1)$ is the probability of first leaving state $x_\pm$ at time $t_2$ given that the state was entered at time $t_1$, and $Z_\pm(t)$ is the probability that the state was entered at time $t$.

For any periodic modulation of the transition rates, the power spectrum $S(\omega)$ is the Fourier transform of a temporal correlation function $C(\tau)$, which is the sum of "signal" and "noise." We show that the signal is periodic in time with period $T_s$, and the noise is the product of an exponential decay with a (different) periodic function with period $T_s$ [10]. The residence-time distribution $V_\pm(\tau)$ is the product of an exponential envelope and a function that is periodic with the drive period $T_s$. The decay rate of the exponential is just the average of the relevant rate over the drive period. We show that calculating both the power spectrum and the residence-time distribution nonperturbatively can be done by integrating periodic functions over a single drive period (though obtaining the residence-time distribution involves iterating a set of self-consistent equations).

The paper is organized as follows. Section II discusses the properties of the power spectrum, Sec. III concerns the residence-time distribution, Sec. IV applies the results to experiments, including those probing a dissipative quantum two-state system, and Sec. V is a summary.

## II. POWER SPECTRUM

This section addresses the power spectrum characterizing the system's response, which is the Fourier transform of the temporal correlation function $C(\tau) \equiv \langle x(t+\tau)x(t)\rangle$, defined in Eq. (1.2). It is shown here that $C(\tau)$ is the sum of a signal $C_S(\tau)$, which is periodic, $C_S(\tau+T_s)=C_S(\tau)$, and a noise term $C_N(\tau)$, which can be written $C_N(\tau)=e^{-\langle W\rangle\tau}\chi_N(\tau)$, where $\chi_N(\tau)=\chi_N(\tau+T_s)$. Here, $\langle W\rangle$ is the time average of the sum of the rates $\langle W\rangle = \langle (W_-(t)+W_+(t))\rangle$. In Appendix B the power spectrum $S(\omega)$ is expressed in terms of the Fourier components of $C_S(\tau)$ and $\chi_N(\tau)$.

We first separate $C(\tau)$ into signal and noise components. The starting point is the expression for $C(\tau)$ in terms of the conditional probabilities $n_+(t_2|x_1,t_1)$, Eq. (1.2). It is shown in Appendix A that $n_+(t_2) \equiv \lim_{t_2-t_1\to\infty} n_+(t_2|x_1,t_1)$ is independent of both $x_1$ and $t_1$ and periodic with period $T_s$. By defining $\delta n_+(t+\tau|\pm,t) \equiv n_+(t+\tau|\pm,t) - n_+(t+\tau)$, one can write $C(\tau)=C_S(\tau)+C_N(\tau)$, where

$$C_S(\tau) = \langle [1-2n_+(t+\tau)][1-2n_+(t)]\rangle , \quad (2.1a)$$

$$C_N(\tau) = 2\langle [\delta n_+(t+\tau|+,t)+\delta n_+(t+\tau|-,t)]n_+(t) - \delta n_+(t+\tau|-,t)\rangle . \quad (2.1b)$$

The calculation of the signal proceeds using the result for $n_+(t)$ given in Appendix A:

$$n_+(t) = \frac{1}{1-e^{-\langle W\rangle T_s}} \int_0^{T_s} dt^* W_-(t-t^*) e^{-\langle W\rangle t^*} \times h(t-t^*,t) , \quad (2.2)$$

where

$$h(t_1,t_2) = \exp\left[-\int_{t_1}^{t_2} dt^* \delta W(t^*)\right] , \quad (2.3)$$

$W(t)=W_+(t)+W_-(t)$, and $\delta W(t)=W(t)-\langle W\rangle$. Since $W_-(t_1+T_s)=W_-(t_1)$ and $h(t_1+T_s,t_2)=h(t_1,t_2+T_s) = h(t_1,t_2)$, it follows that $n_+(t+T_s)=n_+(t)$. Therefore, $n_+(t)$ can be expanded in a Fourier series [11],

$$n_+(t) = \sum_{m=-\infty}^{\infty} \eta_+(m)e^{im\omega_s t} , \quad (2.4)$$

and the correlation function $C_S(\tau)$ is obtained using Eq. (2.1a):

$$C_S(\tau) = 1-4\eta_+(m=0)+4\sum_{m=-\infty}^{\infty}|\eta_+(m)|^2 e^{im\omega_s\tau} \equiv \sum_{m=-\infty}^{\infty}\tilde{C}_S(m)e^{im\omega_s t} . \quad (2.5)$$

Clearly, $C_S(\tau)$ is periodic in the drive period $T_s = 2\pi/\omega_s$. Using the nonperturbative solution of Eqs. (2.2) and (2.3) for $n_+(t)$ derived in Appendix A [Eq. (A5)], using Eq. (2.5) one can obtain $C_S(\tau)$ for any periodic time dependence of the rates.

For a small amplitude drive of the form $W_\pm(t)=\langle W_\pm\rangle + w_\pm\cos(\omega_s t)$, one can expand the expression for $n_+(t)$, Eqs. (2.2) and (2.3), in powers of $w_\pm$. In the absence of modulation, $n_+=W_-/W$. If the rates obey detailed balance ($W_+/W_-=e^{\beta\varepsilon_0+\delta(\beta\varepsilon)\cos(\omega_s t)}$), then the modulation induces a contribution to $n_+(t)$ at zero frequency as well as at the drive frequency $\omega_s$. The component at $\omega_s$ leads to a signal $\delta n_+(t)\cong\zeta(\omega_s)\cos(\omega_s t-\phi)$, where

$$\zeta(\omega_s)=-\frac{W_+}{(1+W_+/W_-)}\frac{\delta(\beta\varepsilon)}{\sqrt{W^2+\omega_s^2}}\ , \tag{2.6}$$

and $\phi=\tan^{-1}(\omega_s/W)$. The modulation also causes the time average $\langle n_+\rangle$ to change; $\langle n_+\rangle=W_-/W-\langle\delta n_+\rangle$, where

$$\langle\delta n_+\rangle=\frac{1}{4W}\left[\frac{\partial^2 W_-}{\partial h^2}-\frac{\partial^2 W}{\partial h^2}\frac{W_-}{W}\right](\delta h)^2 -\frac{1}{2}\frac{1}{\omega_s^2+W^2}\left[\frac{\partial W}{\partial h}\right]\times\left[\frac{\partial W_-}{\partial h}-\frac{\partial W}{\partial h}\frac{W_-}{W}\right](\delta h)^2\ . \tag{2.7}$$

Here we have expressed the modulation in terms of a control parameter $h$; $h=T$ for temperature driving and $h=\varepsilon$ for asymmetry modulation. Although the leading-order signal component at $\omega_s$ depends only on the variation in $\varepsilon/k_BT$, $\langle\delta n_+\rangle$ depends explicitly on the variation of the rates themselves.

Equations (2.5) and (2.7) yield the result for $C_S(\tau)$ to order $\delta h^2$:

$$C_S(\tau)=\left[1-2\frac{W_-}{W}\right]\left[\left[1-2\frac{W_-}{W}\right]-4\langle\delta n_+\rangle\right]+2\zeta^2(\omega_s)\cos(\omega_s\tau)\ . \tag{2.8}$$

The calculation of the continuous portion of the power spectrum follows from the expression for $C_N(\tau)$ in terms of $\delta n_+$ and $n_+$ (Eq. 2.1b) along with the expressions for $\delta n_+$ and $n_+$ in Appendix A [Eqs. (A5), (A6b), and (A8)]. We find

$$C_N(\tau)=e^{-\langle W\rangle\tau}\frac{4}{T_s}\int_0^{T_s}dt[1-n_+(t)]n_+(t)h(t,t+\tau) \equiv e^{-\langle W\rangle\tau}\chi_N(\tau)\ , \tag{2.9}$$

where $n_+(t)$ is given in Eq. (2.2) and $h(t_1,t_2)$ is defined in Eq. (2.3).

In the absence of modulation, the noise contribution to the power spectrum at $\omega_s$ is

$$S_N(\omega_s)|_0=\frac{4W_+W_-}{W}\frac{1}{W^2+\omega_s^2}\ . \tag{2.10}$$

Perturbative corrections to $S_N(\omega_s)$ can be obtained straightforwardly by expanding Eq. (2.9).

The SNR is the ratio of the weight in the $\delta$-function peak at $\omega_s$ to the amplitude of the broadband noise background at this frequency. When the modulation is small, the leading contribution to the signal is of order $(\delta(\beta\varepsilon))^2$. The $\delta$-function peak in the power spectrum at $\omega_s$ is calculated using Eqs. (2.8) and (B5); it has magnitude $\pi\zeta^2(\omega_s)$. When the modulation is small, corrections to the noise amplitude, which are of order $(\delta(\beta\varepsilon))^2$, do not contribute to the leading term of the SNR. Thus, Eqs. (2.8) and (2.10) yield the lowest order contribution to the SNR, $\mathcal{S}$

$$\mathcal{S}\cong\frac{\pi}{4}\frac{W_+W_-}{W}[\delta(\beta\varepsilon)]^2\ . \tag{2.11}$$

This result agrees with that found previously for a classical system characterized by activated transition rates [2,8]. Linear response theory has been used to derive (2.11) for both asymmetry and temperature modulation of a classical system [3]. Nonperturbatively, the SNR can be obtained by dividing the signal and noise obtained using the nonperturbative results in this section.

We apply our results to calculate nonperturbatively the power spectra of two-state systems subjected to temperature driving. In the nonperturbative regime the signal can be obtained using Eqs. (2.2)–(2.8). Numerical integration is facilitated if one uses a Fourier expansion of $W(t)$ and integrates (2.3) analytically to obtain $h(t_1,t_2)$. The resulting expression is a sum of single integrals over one drive period, and thus is straightforward to evaluate numerically.

First we consider activated rates $W_+=\omega_0 e^{-U/k_BT(t)}$, $W_+/W_-=e^{\varepsilon/k_BT(t)}$, where $T(t)=T_0+\delta T\cos(\omega_s t)$. The transition rates $W_+(t)$ and $W_-(t)$ are shown in Fig. 2; for the parameter values used (given in the figure) the modulations of the rates are large compared to their average values. The resulting signal $n_+(t)$ has significant harmonic content; the component at $2\omega_s$ is nearly one-fourth as large as the fundamental [12]. The noise $S_N(\omega)$ for the same parameters is shown in Fig. 3; even though the first harmonic of $\tilde{\chi}_N(m)$ is about 10% of the fundamental at $m=0$, the noise is barely distinguishable from a simple Lorentzian.

The transition rates for the two-state system undergoing quantum tunneling in the presence of an Ohmic bath [13] are discussed in Appendix C; the two transition rates

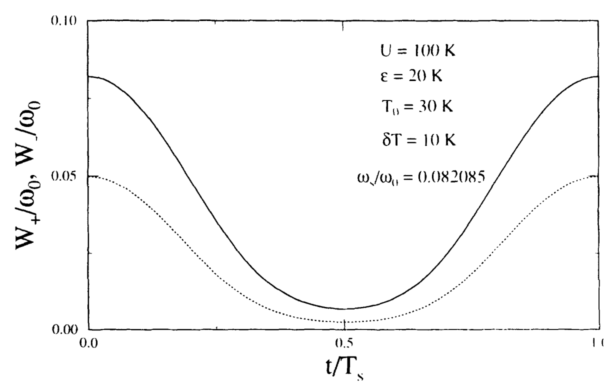


FIG. 2. Fast rate $W_+(t)$ and slow rate $W_-(t)$ for a two-state system with activated transition rates, $W_+=\omega_0 e^{-U/k_BT(t)}$, $W_+/W_-=e^{\varepsilon/k_BT(t)}$, with $T(t)=T_0+\delta T\cos\omega_s t$, for parameter values given in the figure.

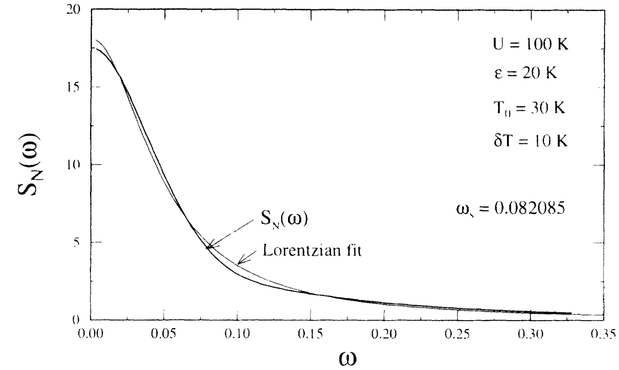


FIG. 3. Noise $S_N(\omega)$ for a two-state system with activated transition rates, for parameter values given in the figure. The parameter $\omega_0$ has been set to unity.

are related by detailed balance, and the dependence of the normalized fast rate $W_+$ for a given asymmetry ε on the temperature $T$ is given by

$$\frac{W_+(T)}{W_+(T_0)}=\left[\frac{T}{T_0}\right]^{-(1-2\alpha)} e^{\varepsilon/2k_B(1/T-1/T_0)} \times\frac{\left|\Gamma\left[\alpha+\frac{i\varepsilon}{2\pi k_B T}\right]\right|^2}{\left|\Gamma\left[\alpha+\frac{i\varepsilon}{2\pi k_B T_0}\right]\right|^2}, \tag{2.12}$$

where $T_0$ is a reference temperature. As discussed in Appendix C, $\alpha$ is a parameter that characterizes the coupling between the bath and the two-state system with asymmetry ε. We scale the time so that $W_+(T_0=1K)=1$.

Figure 4 shows the normalized transition rates as a function of temperature below 1 K for $\alpha=0.25$ and $\varepsilon=0.7$ K. The fast rate $W_+$ depends only weakly on temperature and asymmetry. In the limit of a symmetric well, $\varepsilon/k_BT\ll 1$, $W_\pm \propto T^{-(1-2\alpha)}$ monotonically increases as $T$ decreases. Equation (2.11) yields the normalized SNR for this system in the limit of small modulation; the scaled SNR's for asymmetry and temperature driving are shown in Fig. 5. Unlike the classical case where modulation of a symmetric well can yield a maximum in the SNR as a function of temperature (e.g., for asymmetry driving [2,3,6,8]), the quantum case has a peak only when the well is asymmetric [9]. When $\varepsilon_0/k_BT\gg 1$, the signal is suppressed by the exponentially small slow transition rate, i.e., the particle does not leave the lower well. When $k_BT\sim\varepsilon$ the relative occupation in the upper state depends more sensitively on temperature; when $k_BT\gg\varepsilon$ the relative occupations are nearly equal and the signal again decreases.

Figure 6 shows the numerically evaluated Fourier components of the signal $\tilde{C}_S(\omega)$ [the Fourier transform of

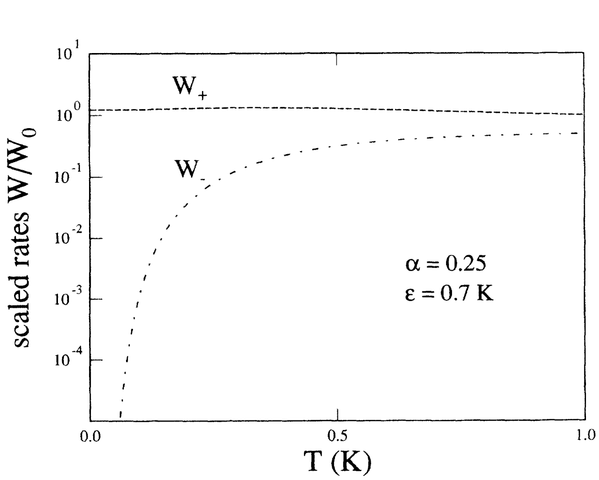


FIG. 4. Transition rates $W_+$ and $W_-$ of a quantum-mechanical dissipative two-state system vs temperature $T$. The rates are scaled by $W_0\equiv W_+(T=1$ K).

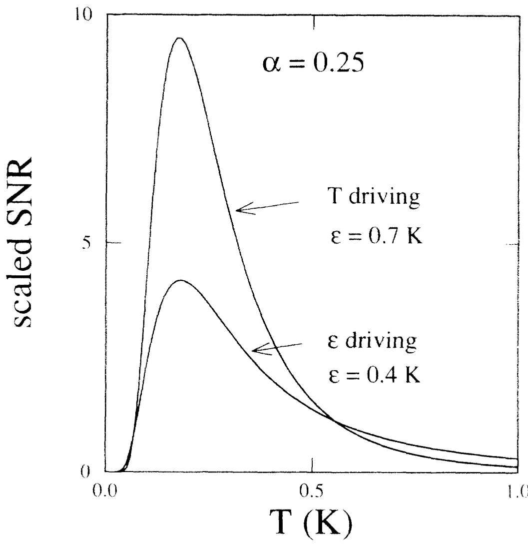


FIG. 5. Signal-to-noise ratio for a quantum-mechanical dissipative two-state system in the limit of small modulation [Eq. (2.11)] vs temperature $T$. The SNR is scaled by $W_0(\delta h)^2$, where $W_0\equiv W_+(1$ K) and $\delta h$ is $\delta T(\delta\varepsilon)$ for temperature (asymmetry) modulation.

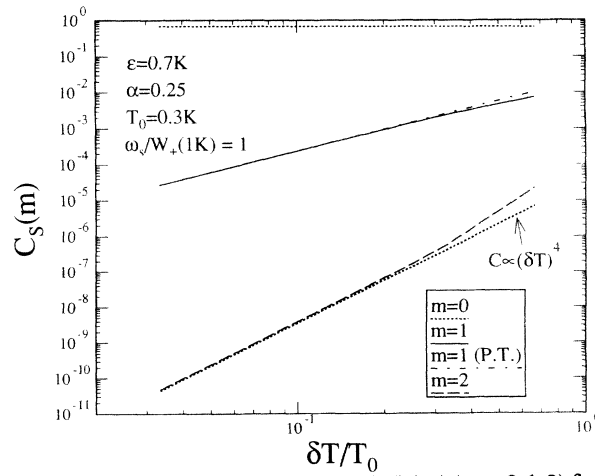


FIG. 6. Fourier coefficients of signal $C_S(m)$ ($m=0,1,2$) for a modulated dissipative quantum two-state system with parameter values $\alpha=0.25$, $\varepsilon=0.7$ K, $T_0=0.3$ K, and different values of $\delta T$. The perturbative result for the $m=1$ component is also shown.

$C_S(\tau)$] at frequencies $\omega=0$, $\omega=\omega_s$, and $\omega=2\omega_s$, for temperature modulation of the form $T(t)=T_0+\delta T\cos(\omega_s t)$, for different values of $\delta T$. For comparison, we have also plotted the leading-order perturbative result for the component at $\omega_s$. The nonperturbative corrections appear to be rather small, and the perturbation theory works reasonably well even when $\delta T/T_0=\frac{2}{3}$. We have also plotted the component at $2\omega_s$ as a function of $\delta T/T_0$. This harmonic is always substantially smaller than the leading correction, and it depends on the drive as $(\delta T)^4$ over a substantial range of $\delta T$. This dependence is consistent with that obtained using naive power-counting arguments for the perturbation series. The noise $C_N(\tau)$ is calculated using Eq. (2.9); for the parameter values used for Fig. 6 the power spectrum $S_N(\omega)$ is indistinguishable from a Lorentzian on a plot, so we do not display it explicitly. Thus, in both the classical and quantum limits, the perturbation theory provides a good qualitative description of the behavior even when the modulation is not small.

## III. RESIDENCE-TIME DISTRIBUTION

The residence-time distribution $V_\pm(\tau)$ describes the probability that the system remains in state $x_\pm$ for a duration $\tau$. In this section we show that $V_\pm(\tau)$ can be written as the product of an exponential and a periodic function, and we show how to calculate it nonperturbatively.

The residence-time distribution obeys

$$V_\pm(\tau)=N\int_{-\infty}^{\infty}dt_0 P_\pm(t_0+\tau|t_0)Z_\pm(t_0)\ , \tag{3.1}$$

where $N$ is a normalization constant, $P_\pm(t_2|t_1)$ is the probability of first leaving the $\pm$ state at time $t_2$ given that the state was entered at time $t_1$, and $Z_\pm(t)$ is the probability that the state was entered at a time $t$. The normalization of $Z_\pm(t)$ is arbitrary; we choose it so that $\int_{-\infty}^{\infty}dt\, Z_\pm(t)=1$; the normalization constant $N$ for $V_\pm(\tau)$ is fixed by the condition $\int_0^\infty d\tau\, V_\pm(t)=1$. We define $Y_\pm(t^*)$ to be the probability that the state was entered at a time $t^*$ satisfying $t^*=t\bmod(T_s)$: $Y_\pm(t)=\sum_{m=-\infty}^{\infty}Z_\pm(t+mT_s)$. By definition, $Y_\pm(t)=Y_\pm(t+T_s)$ and $\int_0^{T_s}dt\, Y_\pm(t)=1$, and using (3.1) it is straightforward to show that

$$V_\pm(\tau)=N\int_{-\infty}^{\infty}dt_0 P_\pm(t_0+\tau|t_0)Y_\pm(t_0)\ . \tag{3.2}$$

If the system enters a state at time $t_1$, in order to remain until a time between $t_2$ and $t_2+\delta t$, it must first remain in the state for times less than $t_2$, and then leave the state between $t_2$ and $t_2+\delta t$. Thus,

$$P_\pm(t_2|t_1)=W_\pm(t_2)\exp-\int_{t_1}^{t_2}dt^*W_\pm(t^*)\ . \tag{3.3}$$

By defining $\langle W_\pm(t)\rangle\equiv W_\pm(t)-\delta W_\pm(t)$, it follows that $\int_{t_1}^{t_1+T_s}dt^*\delta W_\pm(t)=0$. Since $W_\pm(t+T_s)=W_\pm(t)$,

$$P_\pm(t_0+\tau+T_s|t_0)=e^{-\langle W_\pm(t)\rangle T_s}P_\pm(t_0+\tau|t_0)\ . \tag{3.4}$$

Therefore, $V_\pm(\tau)$ is the product of an exponential envelope and a function that is periodic with the drive period $T_s$,

$$V_\pm(\tau)=Ne^{-\langle W_\pm\rangle\tau}G_\pm(\tau)\ , \tag{3.5}$$

where $G_\pm(\tau+T_s)=G_\pm(\tau)$.

To calculate the function

$$G_\pm(\tau)=\int_0^{T_s}dt_0 W_\pm(t_0+\tau)\times\exp\left[-\int_{t_0}^{t_0+\tau}dt^*\delta W_\pm(t^*)\right]Y_\pm(t_0)\ , \tag{3.6}$$

one must determine $Y_\pm(t_0)$. At long times the probability of arrival relative to the phase of the drive tends to a fixed distribution, so $Y_\pm$ satisfies

$$Y_\mp(t)=\int_{-\infty}^{t}dt_1 P_\pm(t|t_1)Y_\pm(t_1)\ . \tag{3.7}$$

Using Eqs. (3.3) and (3.4), this integral can be recast into the form

$$Y_\mp(t)=\frac{W_\pm(t)}{1-e^{-\langle W_\pm\rangle T_s}}\times\int_0^{T_s}dt_1 Y_\pm(t-t_1)\times e^{-\langle W_\pm\rangle t_1}\exp\left[-\int_{t-t_1}^{t}dt^*\delta W_\pm(t^*)\right]\ . \tag{3.8}$$

Equation (3.8) involves only integrals over finite time intervals. If one integrates $\int_{t-t_1}^{t}dt^*\delta W_\pm(t^*)$ analytically by expanding $\delta W_\pm(t)$ in a Fourier series, only a single numerical integration over one drive period is necessary. These equations can be solved by iteration; for instance, one can start with the function $Y_+(t)=\text{const}$, and then apply Eq. (3.8) repeatedly until the output function no longer changes as the equation is iterated.

Once $Y_\pm(t)$ is obtained, it is straightforward to use Eqs. (3.5) and (3.6) to obtain $V_\pm(\tau)$. Once again, one need integrate numerically over only one period of the drive.

One can use (3.2)–(3.8) to calculate $G_\pm(\tau)=G_{\pm 0}+\delta G_\pm(\tau)$ perturbatively for $\delta W_\pm(t)=w_\pm\cos(\omega_s t)$, when $w_\pm$ is small. In the limit $w_\pm=0$ the transitions are completely uncorrelated with the drive, and both $Y_\pm(t)$ and $G_\pm(t)$ are independent of $t$. We choose the normalization of $V_\pm(\tau)$ so that $\int_0^\infty V_\pm(\tau)=1$; thus in the undriven system $G_{\pm 0}(\tau)=W_\pm$.

The effect of the drive is to enhance and suppress the transition probability periodically. As a result $Y_\pm(t)$ acquires a periodic time dependence, i.e., $Y_\pm(t)=Y_{0\pm}+\sum_1^\infty\delta Y_{n\pm}\cos(n\omega_s t+\delta_{n\pm})$. Two successive applications of Eq. (3.8) yield a self-consistency equation for $Y_\pm(t)$. One can evaluate the unknown factors in the expression for $Y_\pm(t)$ by expanding the self-consistent equation in powers of $w_\pm$. The leading-order correction to $Y_\pm(t)$ is proportional to $w_\pm$ and at the drive frequency. Using this in Eq. (3.6), one finds the leading term in $\delta G_\pm(\tau)$, proportional to $w_\pm^2$. Two interesting limiting cases are

$\omega_s/2\pi \gg W_+, W_-$, where to lowest order,

$$\frac{\delta G_\pm(\tau)}{G_{\pm 0}} = \frac{1}{2}\frac{w_-}{W_{-0}}\frac{w_+}{W_{+0}}\cos\omega_s\tau \;, \tag{3.9a}$$

and $W_- \ll \omega_s/2\pi \ll W_+$, $\delta W_+ \to 0$, where $G_+(\tau) = W_+$, and

$$\frac{\delta G_-(\tau)}{G_{-0}} = \frac{1}{2}\frac{w_-^2}{W_{-0}^2}\cos\omega_s\tau \;. \tag{3.9b}$$

In the slow modulation limit $W_\pm T_s \ll 1$ the exponential falloff in (3.5) dominates and $V(\tau)$ has no peaks.

In the limit of low temperatures ($\varepsilon/k_B T \gg 1$), the slow rate $W_-$ is exponentially suppressed and thus extremely sensitive to temperature (or asymmetry) variation. This sensitivity implies that the periodically modulated $W_-$ can be represented as a sum of $\delta$ functions at times $t_0 + mT_s$ where $\varepsilon/k_B T$ is a minimum, i.e., $W_-(t) = \eta_- \sum_{m=-\infty}^{\infty} \delta(t - t_0 - mT_s)$, where $\eta_- = \langle W_- \rangle T_s$. In the classical limit, $W_+(t)$ is also strongly temperature dependent, so $W_+(t) = \eta_+ \sum_m \delta(t - t_1 - mT_s)$, where $t_1 = t_0 + T_s/2$ for $\varepsilon$ driving and $t_1 = t_0$ for $T$ driving. The self-consistency condition for $Y_\pm(t)$ obtained by two applications of (3.8) is satisfied by $Y_+(t) \propto \delta(t - t_0)$ and $Y_-(t) \propto \delta(t - t_1)$. Using these expressions in (3.6), one finds that for $\varepsilon$ driving $G_\pm(\tau)$ have maxima at $\tau = T_s/2$ and for $T$ driving, $G_\pm(\tau)$ have maxima when $\tau = T_s$.

In the quantum case $W_+$ is approximately temperature independent. Using $W_+ = \text{const}$ and $W_- = \eta_- \sum_m \delta(t - mT_s)$, and (without loss of generality) taking $t_0 = 0$, one finds that the self-consistent solution to Eq. (3.8) is $Y_+(t) \propto \delta(t)$ and $Y_-(t) \propto e^{-\langle W_+ \rangle t}$. $G_+(\tau)$ is a constant in this limit. When $\langle W_+ \rangle t \gg 1$, $Y_-(t)$ is sharply peaked near $t = 0$, and from (3.6) one finds that $G_-(\tau)$ has peaks near $\tau = T_s$.

The general behavior suggested by (3.9a) and (3.9b) agrees with results of analog simulations of systems with classical rates [7]. For a symmetric well with thermally activated rates, the residence-time distributions have been previously analyzed in the perturbative and low-temperature limits [5,14].

We obtain nonperturbative results for $V_\pm(\tau)$ by numerical solution of Eqs. (2.2)–(2.8). Qualitatively, the behavior interpolates smoothly between the perturbative results and the low temperature, large modulation limit. For activated rates, $V_\pm(\tau)$ can display peaks at half-integral multiples of the drive period if one modulates the asymmetry (Fig. 7) [15] and near integral multiples of $T_s$ if the temperature is modulated (not shown). For the quantum case, modulating the asymmetry and the temperature lead to qualitatively similar results. Figure 8 shows $V_-(\tau)$ evaluated at the peak of the SNR curve for $T$ driving shown in Fig. 5 for two frequencies. The fast modulation limit yields very small harmonic content because $w_+$ is very small. However, in the limit $W_- \ll 1/T_s \ll W_+$, $G_-(\tau)$ is a maximum and $V_-(\tau)$ displays peaks at values $\tau = nT_s$. The residence time distribution for the upper well $V_+(\tau)$ displays little structure for any drive frequency because $W_+$ is nearly con-

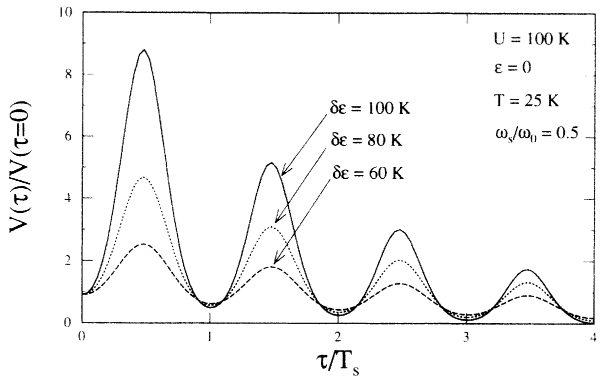


FIG. 7. Distribution of residence times $V(\tau)$ vs $\tau/T_s$, obtained by numerical solution of Eqs. (3.5)–(3.8) for a two-state system with activated transition rates $W_+(t) = \omega_0 e^{-[U-\varepsilon(t)/2]/k_B T}$, $W_-(t) = e^{-\varepsilon(t)/k_B T} W_+(t)$, with asymmetry modulation $\varepsilon(t) = \varepsilon_0 + \delta\varepsilon \cos\omega_s t$. $V_+(\tau) = V_-(\tau)$ because $\varepsilon_0 = 0$; $V(\tau)$ is normalized to unity at $\tau = 0$. Three different values of the asymmetry modulation $\delta\varepsilon$ are shown; parameter values are given in the figure. The residence-time distribution has maxima near half-integral multiples of the drive period $T_s$.

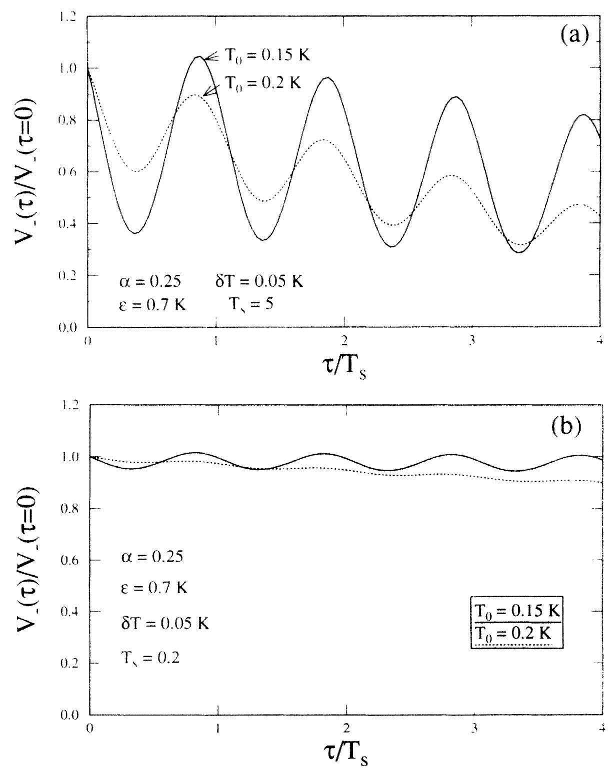


FIG. 8. Distribution of residence times in the lower energy well $V_-(\tau)$ vs $\tau/T_s$, obtained by numerical solution of Eqs. (3.5)–(3.8) for a two-state system undergoing dissipative quantum tunneling with temperature driving $T(t) = T_0 + \delta T \cos 2\pi t/T_s$. $V_-(\tau)$ is normalized to unity at $\tau = 0$. (a) $T_s = 5$. (b) $T_s = 0.2$. Parameter values are given in the figure; the units of time are normalized so that $W_+(T = 1\text{ K}) = 1$. $V_-(\tau)$ displays large oscillations with period $T_s$ when $W_- \ll 1/T_s \ll W_+$. Two different values of $T_0$ are shown; as $T_0$ is lowered the oscillations in $V_-(\tau)$ become more pronounced.

stant in the quantum system. Note that even in lowest order $V_{\pm}(\tau)$ depends explicitly on the temperature dependence of the rates, so that it differs fundamentally from the SNR. In the fast modulation limit $1/T_s \gg W_{\pm}$ the quantum system displays a maximum in the SNR but very little structure in $V_{\pm}(\tau)$. Therefore, these two quantities probe different aspects of the dynamics.

## IV. EXPERIMENTAL OBSERVABILITY

Here we estimate the counting time needed to resolve the SNR and the structure in the residence-time distribution, given the time-dependent rates $W_{\pm}(t)$. The theory presented here assumed that the system has only two states, so that $x$ takes on only two values ($x_+ = +1$ and $x_- = -1$); noise arises only because transitions between the two states are governed by stochastic equations. In a real experiment, one expects to observe noise even if the system remains nominally in a single state. All of our calculations assume that the noise is dominated by the stochastic nature of the switching times, so that individual switches can be resolved.

Previously we estimated $\mathcal{T}$, the time needed to resolve the signal, using the SNR ratio itself [9]. The SNR $\mathcal{S}$, has units of frequency because the signal provides $\delta$-function contributions to the power spectrum, whereas the noise is a continuous function of frequency. In a time series of duration $\mathcal{T}$ the minimum frequency bin has width $\Delta\omega \sim 2\pi/\mathcal{T}$. One can only resolve the signal if the noise power in this frequency window is less than the weight of the signal power contained in the $\delta$ function. Thus, $\mathcal{T}$ must be large enough that

$$\mathcal{T} \gg \frac{2\pi}{\mathcal{S}} . \tag{4.1}$$

The same estimate can be obtained by applying simple statistical considerations to the experimentally observed "telegraph" signal characterizing the transitions between the two states.

We consider the limit of small modulation, so that the signal can be obtained using the perturbative expression [Eq. (2.6)] for $n_+(t)$. We then ask how long one must take data so that random uncertainties inherent to measurements of the unmodulated system are smaller than this signal. We consider the cases $\omega_s \lesssim W$ and $\omega_s \gtrsim W$ separately.

We first consider the case of slow driving, $\omega_s \lesssim W$. We resolve the temporal variation in $n_+(t)$ by dividing up each drive period $T_s$ into $M$ bins and measuring the changes in the number of observations of the + state as the bin index $j$ is varied. Because the correlation between successive measurements decays with time constant $1/W$, in a time $\mathcal{T}$ one can make $N_{\rm trial} = W\mathcal{T}/M$ uncorrelated measurements in each bin of the system's state. Without modulation the system is in state $\pm$ with probability $W_{\mp}/W$. Therefore, the chance of obtaining $N_+$ +'s out of $N_{\rm trial}$ trials is

$$\mathrm{Prob}(N_+) = \frac{N_{\rm trial}!}{N_+!(N_{\rm trial}-N_+)!} \left[\frac{W_-}{W}\right]^{N_+} \times \left[\frac{W_+}{W}\right]^{(N_{\rm trial}-N_+)} \tag{4.2}$$

The binomial distribution (4.2) implies that the expectation value of $N_+$ in a bin is $(W_-/W)N_{\rm trial}$, and that this value will exhibit statistical fluctuations of size $\sim[N_{\rm trial}(W_+W_-/W^2)]^{1/2}$.

The signal can be resolved if the modulation induces a variation in $N_+$ that is larger than the statistical uncertainty. This criterion leads to the condition that

$$\left[\frac{W\mathcal{T}}{M}\right]\frac{W_+W_-}{W}\frac{\delta\varepsilon}{\sqrt{W^2+\omega_s^2}} \gtrsim \left[\frac{W_-W_+\mathcal{T}}{M}\right]^{1/2} , \tag{4.3}$$

so that

$$\mathcal{T} \gtrsim \frac{MW}{W_+W_-}\frac{(1+\omega_s^2/W^2)}{(\delta(\beta\varepsilon))^2} . \tag{4.4}$$

This result agrees with Eq. (4.1) when $\omega_s \lesssim W$.

When the sum of the rates $W$ is small compared to the drive frequency $\omega_s$, then the estimate (4.4) yields a significantly longer time than (4.1). However, when $\omega_s \gg W$, the signal can be resolved more efficiently by measuring $dn_+/dt$ rather than $n_+(t)$ itself. We once again divide each drive period $T_s$ into $M$ bins and measure, as a function of bin index $j$, the quantity $R_+(j,\mathcal{T}) - R_-(j,\mathcal{T})$, where $R_{\pm}(j,\mathcal{T})$ is the number of transitions out of the $\pm$ state in the $j$th bin observed in a counting time $\mathcal{T}$. The probability of a transition out of the $\pm$ state in a time interval $\delta t$ is $n_{\pm}(j)W_{\pm}(j)\delta t$. Thus, the expected number of transitions in the $j$th bin given an observation time $\mathcal{T}$ is

$$\overline{R_{\pm}}(j,\mathcal{T}) = \frac{\mathcal{T}}{M} n_{\pm}(j) W_{\pm}(j) \tag{4.5a}$$

$$\cong \frac{\mathcal{T}}{M}\left[\left[\frac{W_{\mp}}{W}\right]W_{\pm} + \left[\frac{W_{\mp}}{W}\right]\delta W_{\pm}(j) + W_{\pm}\delta n_{\pm}(j)\right] , \tag{4.5b}$$

where Eq. (4.5b) applies in the limit of weak modulation and we have used the fact that in the unmodulated system, $n_{\pm} = W_{\mp}/W$. The modulation leads to a difference in the expectation value

$$\overline{R_+}(j,\mathcal{T}) - \overline{R_-}(j,\mathcal{T}) = \frac{\mathcal{T}}{M}\left[\frac{\delta W_+}{W_+} - \frac{\delta W_-}{W_-} + W\delta n_+\right]\frac{W_+W_-}{W} = \frac{\mathcal{T}}{M}\frac{W_+W_-}{W}\delta(\beta\varepsilon)\left[1 - \frac{W}{\sqrt{W^2+\omega_s^2}}\right] . \tag{4.6}$$

To resolve this difference, it must be larger than the $\sqrt{N}$ fluctuations in the absence of driving, leading to the condition

$$\overline{R_+}(j,\mathcal{T}) - \overline{R_-}(j,\mathcal{T}) \gtrsim \left[\frac{\mathcal{T}}{M}\frac{W_+W_-}{W}\right]^{1/2} , \tag{4.7}$$

from which one obtains

$$\mathcal{T} \gtrsim \frac{MW}{W_+ W_- (\delta(\beta\varepsilon))^2 [1 - W/\sqrt{W^2+\omega_s^2}]^2} . \quad (4.8)$$

This condition is equivalent to (4.1) when $\omega_s \gg W$.

The expected number of switches out of each state in a time $\mathcal{T}$ is $W_+ W_- \mathcal{T}/W$, so the condition (4.1) on counting times can be written in the weak modulation limit as

$$N_{\text{switch}} \gtrsim \frac{4\pi}{(\delta(\beta\varepsilon))^2} , \quad (4.9)$$

where $N_{\text{switch}}$ is the total number of switches in the observation time.

The maximum SNR attainable for a given $\delta T$ or $\delta\varepsilon$ increases as the lowest experimentally available temperature $T_{\min}$ decreases. For the dissipative tunneling of a defect in mesoscopic Bi samples (as described in Appendix C), $T_{\min}$ is limited to 0.1 K because of Ohmic heating. For reasonable experimental parameters [16], $T_{\min} \sim 0.1$ K, $\varepsilon \sim 0.4$ K, $\delta\varepsilon \sim 0.05$ K, and $W_+ \sim 10$ Hz, a maximum SNR of 10 (at 0.2 K) requires $\mathcal{T} \sim 1100$ s. For temperature driving with $\delta T \sim 0.05$ K and $\varepsilon \sim 0.7$ K, resolving a SNR of 10 takes 360 s. These times are reasonable since typical measurement intervals are 200–2000 s long.

The counting times required to resolve structure in the residence-time distribution restrict the drive frequency. If the drive frequency is too small compared to the transition rate in question, the exponential envelope of the residence-time distribution function causes there to be a very small fraction of counts at $\tau \gtrsim T_s$. However, the drive period must be long enough that a reasonable fraction of the residence times are in the interval between 0 and $T_s$. The drive period also affects the magnitude of the structure in $V(\tau)$ that one is trying to resolve. In particular for the quantum case, the oscillations are much larger if $W_+ \gg 1/T_s$ than when $W_+ \lesssim 1/T_s$.

The counting time required to resolve structure in $V(\tau)$ can be estimated by comparing the theoretical amplitude of the oscillations to the statistical fluctuations inherent in the experiment. The uncertainties in number of residence times with values of $\tau$ in bins of width $T_s/M$ are given by $\sqrt{N}$ fluctuations in the number of counts in each bin. For the parameters of Fig. 8 with $W_+ \sim 10$ Hz and $T_0 = 0.2$ K the structure in $V_-(\tau)$ is well resolved in $\lesssim 1000$ s.

## V. SUMMARY

In this paper we have obtained exactly the power spectrum and residence-time distribution characterizing a two-level system whose dynamics are described using time-dependent transition rates. We have applied our results to calculate the power spectrum and the residence-time distribution both for a system described by activated transition rates and for a two-state tunneling undergoing dissipative quantum tunneling.

In the adiabatic regime considered in this paper, the power spectrum is the sum of a signal, which has the same temporal periodicity as the drive, and a noise component, which is the product of an exponential and a periodic function. This simple form for the power spectrum even in the case of strong driving indicates that $\delta$-function subharmonics in the power spectrum can appear only if the adiabatic approximation has broken down. This result may be a useful tool in trying to elucidate the underlying dynamics of driven noisy systems.

## ACKNOWLEDGMENTS

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## APPENDIX A: EVALUATION OF $n_+(t|x_0,t_0)$

In this appendix the conditional probability that the system is in state (+), given that it was in state $x_0$ at time $t_0$, $n_+(t|x_0,t_0)$, is evaluated so that the properties used in Sec. II are apparent. It will be shown that $n_+(t) \equiv \lim_{t_0 \to -\infty} n_+(t|x_0,t_0)$ is independent of both $x_0$ and $t_0$ and periodic in the drive period $T_s$. Moreover, the quantity

$$\delta n_+(t+\tau|x_0,t) \equiv n_+(t+\tau|x_0,t) - n_+(t+\tau)$$

is the product of an exponential $e^{-\langle W\rangle\tau}$ and a function periodic in the drive period. These functions can be written explicitly in terms of integrals over a single drive period.

McNamara and Wiesenfeld [2] present the solution of Eq. (1.1) for $n_+(t)$ given a value $n_+(t_0)$. Defining $W(t) = W_-(t) + W_+(t)$, this solution can be written

$$n_+(t) = \exp\left[-\int_{t_0}^{t} W(t')dt'\right] n_+(t_0) + \exp\left[-\int_{t_0}^{t} W(t')dt'\right] \int_{t_0}^{t} W_-(t') \exp\left[\int_{t_0}^{t'} W(t'')dt''\right] dt' . \quad (A1)$$

Defining $\delta W(t) = W(t) - \langle W\rangle$ and $\delta W_-(t) = W_-(t) - \langle W_-\rangle$, and using the property that

$$\int_{t^*-2T_s}^{t^*-T_s} dt' W_-(t') e^{\langle W\rangle t'} \exp\left[-\int_{t'}^{t} \delta W(t'')dt''\right] = e^{-\langle W\rangle T_s} \int_{t^*-T_s}^{t^*} dt' W_-(t') e^{\langle W\rangle t'} \exp\left[-\int_{t'}^{t} \delta W(t'')dt''\right] \quad (A2)$$

one can recast this into

$$n_+(t) = e^{-\langle W\rangle(t-t_0)} h(t_0,t) n_+(t_0) - \frac{e^{-\langle W\rangle(t-t_0)}}{1-e^{-\langle W\rangle T_s}} \int_0^{T_s} dt_2 W_-(t_0-t_2) e^{-\langle W\rangle t_2} h(t_0-t_2,t)$$
$$+ \frac{1}{1-e^{-\langle W\rangle T_s}} \int_0^{T_s} dt_2 W_-(t-t_2) e^{-\langle W\rangle t_2} h(t-t_2,t) , \quad (A3)$$

where

$$h(t_1,t_2)=\exp\left[-\int_{t_1}^{t_2}\delta W(t^*)dt^*\right] . \quad (A4)$$

Since $\int_{t-T_s}^{t}\delta W(t^*)=0$, for any $t$, it follows that $h(t_0,t+T_s)=h(t_0+T_s,t)=h(t_0,t)$.

In the limit $t-t_0\to\infty$, the first two terms on the right-hand side of Eq. (A3) become exponentially small, and $n_+(t)$ tends to a function independent of the initial condition

$$n_+(t)\equiv \lim_{t-t_0\to\infty} n_+(t|x_0,t_0)$$
$$=\frac{1}{1-e^{-\langle W\rangle T_s}}\int_0^{T_s}dt_2 W_-(t-t_2)\times e^{-\langle W\rangle t_2}h(t-t_2,t) . \quad (A5)$$

Since $W_-$ and $h$ are both periodic functions, it follows that $n_+(t+T_s)=n_+(t)$. Knowledge of $n_+(t)$ is sufficient information to obtain the signal.

To obtain the noise using Eq. (2.1b), one must calculate

$$\delta n_+(t+\tau|+,t)\equiv n_+(t+\tau|+,t)-n_+(t)$$

and

$$\delta n_+(t+\tau|-,t)\equiv n_+(t+\tau|-,t)-n_+(t) .$$

Use of Eq. (A3) yields

$$\delta n_+(t+\tau|-,t)$$
$$=-\frac{e^{-\langle W\rangle\tau}}{1-e^{-\langle W\rangle T_s}}\int_0^{T_s}dt_2 W_-(t-t_2)e^{-\langle W\rangle t_2}\times h(t-t_2,t+\tau) , \quad (A6a)$$

$$\delta n_+(t+\tau|+,t)=\delta n_+(t+\tau|-,t)+e^{-\langle W\rangle\tau}h(t,t+\tau) . \quad (A6b)$$

Finally, because

$$h(t-t_2,t+\tau)=h(t-t_2,t)h(t,t+\tau) , \quad (A7)$$

it follows that

$$\delta n_+(t+\tau|-,t)=-e^{-\langle W\rangle\tau}n_+(t)h(t,t+\tau) . \quad (A8)$$

The expressions Eqs. (A5), (A6b), and (A8) yield $n_+(t)$ and $\delta n_+(t+\tau|x_0,t)$ in quadrature form.

## APPENDIX B: RELATION OF $S(\omega)$ TO PROPERTIES OF $C(\tau)$

In Sec. II it is shown that the correlation function $C(\tau)=\langle x(t+\tau)x(t)\rangle$, where the brackets denote an average over $t$, is the sum of two terms; $C(\tau)=C_S(\tau)+C_N(\tau)$, where $C_S(\tau)$ (the signal) is periodic with period $T_s$, and $C_N(\tau)$ (the noise) can be written

$$C_N(\tau)=e^{-\langle W\rangle\tau}\chi_N(\tau) . \quad (B1)$$

Here, $\langle W\rangle=\langle W_+ + W_-\rangle$ and $\chi_N$ [given in Eq. (2.9)] satisfies $\chi_N(\tau+T_s)=\chi_N(\tau)$ with period $T_s$. Explicit calculation of the functions $C_S$ and $\chi_N(\tau)$ is described in Sec. II. In this appendix the power spectrum $S(\omega)$ is calculated in terms of the Fourier components of $C_S(\tau)$ and $\chi_N(\tau)$. The signal $S_S(\omega)$ is comprised of δ-function peaks at zero frequency, the drive frequency and its harmonics, and a noise $S_N(\omega)$ comprised of the sum of contributions each with width $\langle W\rangle$, centered at zero frequency, the drive frequency, and harmonics of the drive.

These results follow directly if one expands the functions $C_S(\tau)$ and $\chi_N(\tau)$ in a Fourier series and then Fourier transforms $C(\tau)$ to obtain the power spectrum. Thus we define

$$C_S(\tau)=\sum_{m=-\infty}^{\infty}\tilde{C}_S(m)e^{im\omega_s\tau} \quad (B2)$$

and

$$\chi_N(\tau)=\sum_{m=-\infty}^{\infty}\tilde{\chi}_N(m)e^{im\omega_s\tau} . \quad (B3)$$

It follows from Eq. (2.5) that $\tilde{C}_S(m)$ is real and positive for all $m$. We define the power spectrum as

$$S(\omega)=\int_0^{\infty}d\tau\cos(\omega\tau)C(\tau) . \quad (B4)$$

Using Eq. (B2), since $\tilde{C}_S(m)=\tilde{C}_S(-m)$, it follows that

$$S_S(\omega)=2\pi\sum_{m=-\infty}^{\infty}\tilde{C}_S(m)\delta(\omega-m\omega_s) . \quad (B5)$$

For the noise, one finds

$$S_N(\omega)=\int_0^{\infty}d\tau\cos(\omega\tau)e^{-\langle W\rangle\tau}\chi_N(\tau)$$
$$=\frac{1}{2}\sum_{m=-\infty}^{\infty}\left[\frac{\tilde{\chi}_N(m)}{\langle W\rangle-im\omega_s-i\omega}+\frac{\tilde{\chi}_N^*(m)}{\langle W\rangle+im\omega_s+i\omega}\right]$$
$$=\sum_{m=-\infty}^{\infty}\frac{\langle W\rangle\mathrm{Re}\tilde{\chi}_N(m)-\mathrm{Im}\tilde{\chi}_N(m)(m\omega_s+\omega)}{\langle W\rangle^2+(\omega+m\omega_s)^2} . \quad (B6)$$

Finally, we note that because $C(\tau=0)=1$ [17], the coefficients $\tilde{C}_S(n)$ and $\tilde{\chi}_N(n)$ obey a sum rule

$$C(\tau=0)=1=\sum_{n=-\infty}^{\infty}[\tilde{C}_S(n)+\mathrm{Re}(\tilde{\chi}_N(n))] . \quad (B7)$$

## APPENDIX C: TRANSITION RATES FOR A TWO-STATE SYSTEM UNDERGOING DISSIPATIVE TUNNELING

In this appendix we discuss the temperature dependence of the transition rates of a two-state system in the regime where quantum-mechanical tunneling rather than thermal activation dominates. This system is discussed in detail elsewhere [13]; here we summarize a few useful results.

The relevant quantum-mechanical Hamiltonian describes a two-state system coupled to an ensemble of harmonic oscillators [13]:

$$H=\tfrac{1}{2}\varepsilon\sigma_z-\tfrac{1}{2}\hbar\Delta\sigma_x+\sigma_z\sum_\eta V_\eta(b_\eta^\dagger+b_\eta)+\hbar\sum_\eta \omega_\eta\ , \qquad (C1)$$

where $\varepsilon$ is the asymmetry energy, $\Delta$ is the tunneling matrix element, the $\sigma_i$ are Pauli matrices, and $b_\eta^\dagger$ is a harmonic oscillator creation operator with frequency $\omega_\eta$. All necessary information about the effects of the environment is contained in the spectral density $J(\omega)=\pi/2\sum_\eta V_\eta^2\delta(\omega-\omega_\eta)$. Tunneling in metals [18] is described by Ohmic dissipation, $J(\omega)=\alpha(2\pi\hbar\omega)$ for $\omega\ll\Omega_c$, where $\Omega_c$ is a cutoff frequency large compared to $\Delta$ [19].

A parameter determining the dynamics is the renormalized tunneling matrix element $\Delta_r$, related to $\Delta$ by $\Delta_r=\Delta(\Delta/\Omega_c)^{\alpha/(1-\alpha)}$, where $\Omega_c$ is the bath cutoff frequency [20]. If either $\varepsilon$ or $\alpha T$ is much greater than $\hbar\Delta_r$, then the rapid fluctuations of the bath act to dephase the tunneling particle so that the probability of a transition between $x_\pm$ and $x_\mp$ is independent of the system's previous history. The two transition rates obey detailed balance, and the fast rate has the form [21,22]

$$W_+=\frac{\Delta_r^{2-2\alpha}}{2}\left[\frac{2\pi k_B T}{\hbar}\right]^{2\alpha-1}\frac{e^{\varepsilon/2k_BT}}{\Gamma(2\alpha)}\times\left|\Gamma\left[\alpha+\frac{i\varepsilon}{2\pi k_B T}\right]\right|^2\ , \qquad (C2)$$

where $\Gamma$ is the (complex) gamma function. Note that $\Delta_r$ sets the overall scale only, so that normalizing $W_+$ by its value at a reference temperature causes the scaled rate to depend only on $\alpha$, $\varepsilon$, and $T$. Figure 4 shows $W_\pm(T)$ below 1 K scaled by its value at 1 K for $\alpha=0.25$ and $\varepsilon=0.7$ K. $W_+$ is only weakly temperature dependent.

Recent experiments on submicrometer Bi wires have measured transition rates of two-state systems coupled to conduction electrons that below 1 K are well described by Eq. (C2) with values $\alpha\sim0.2$–$0.3$ and $\Delta_r\sim1-5\times10^{-7}$ [16,23]. In these experiments, application of a magnetic field is observed to change the asymmetry energy $\varepsilon$ [24], though apparently not $\alpha$ or $\Delta_r$ [16,23]. Changes in $\varepsilon$ of 0.05 K have been induced by changes in the magnetic field as small as 0.01 T [16]. Therefore, modulation of the asymmetry energy as well as the temperature is possible in this system.

Another system described by Eq. (C1) is the tunneling of flux in a superconducting quantum interference device (SQUID), which can also display transition rates described by Eq. (C2) [25]. Published experimental data [25] are described using $\alpha=1.44$, though smaller values of $\alpha$ have also been obtained [26]. The asymmetry of the potential well depends on the externally applied dc magnetic flux in the loop.

The transition rate description used in this paper has been shown to apply accurately to this system over a broad frequency range. As discussed in detail elsewhere [22,13,27], the description in terms of a two-state system is valid so long as the tunneling matrix element $\Delta_r$ is much smaller than $\sim U/\hbar$, which is expected to be of order hundreds of degrees (e.g., THz). The result that the dynamics are well described by a Markov process holds so long as the tunneling rate is much less than the rate of inelastic collisions between the two-state system and the bath. This condition is satisfied when $\Delta_r/\alpha T\ll1$. In the experiments discussed here, this ratio is of order $10^{-5}$. Therefore, the theory should apply accurately for all frequencies of interest in this system.

We stress that the results in this paper are valid for any transition rates, whether or not a theoretical expression such as (C2) for the rates is available. Experimental determination of the rates provide all the necessary input to apply the theory presented. Therefore, the results can be applied to the tunneling of a defect in the temperature regime above 1 K, where phonon-assisted tunneling [18,28] causes the transition rates to increase as temperature is increased [16,23], as well as to transitions in the activated regime.

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[11] Because $n_+(t)$ is real, these Fourier coefficients obey $\eta_+(m)=\eta_+^*(-m)$.

[12] For these parameters, $\eta_+(1)\approx(5.61-7.25i)\times10^{-3}$, $\eta_+(2)\approx(1.81-1.27i)\times10^{-3}$, $\eta_+(3)\approx(2.06+2.68i)\times10^{-4}$.

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must be corrected to achieve agreement with our results in the classical perturbative regime.

[15] The qualitative behavior of $V_{\pm}(\tau)$ depends on the system parameters; for example, an asymmetric classical system with asymmetry modulation can display peaks at integral multiples of $T_s$ if the parameters are such that $W_{+}(t)$ is much faster than the drive frequency at all times.

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